# COANALYTIC FAMILIES OF FUNCTIONS

JULIA MILLHOUSE AND LUKAS SCHEMBECKER

ABSTRACT. For Van Douwen families, maximal families of eventually different permutations and maximal ideal independent families we show that the existence of a $\Sigma^1_2$ family implies the existence of a $\Pi^1_1$ family of the same size. We also prove a similar, but slightly weaker result for generating sets of cofinitary groups.

## 1. INTRODUCTION

Many combinatorial sets of reals constituting cardinal characteristics can be obtained by an application of the Axiom of Choice or equivalently, a wellordering of the continuum, and hence $\Sigma^1_2$ examples of such sets exist in $\mathsf{L}$, given the $\Sigma^1_2$-definable wellorder of the constructible reals. This was initially observed by Gödel [12]. With more careful methods, the recursive construction can be done in such a way as to yield a coanalytic ($\Pi^1_1$) witness to the combinatorial family in question; this is of particular interest when the family is known to not be analytic as in this case this completely decides the minimal complexity of such an object. A robust coding technique originating in the work of Erdos, Kunen, and Mauldin [5], later streamlined by Miller [17], has been the main tool for obtaining coanalytic witnesses of various combinatorial families in models of $\mathsf{V}$=$\mathsf{L}$, see also [22]. Applied in the literature at the intersection of descriptive set theory and set theory of the reals, we find theorems asserting the consistency of an inequality of cardinal characteristics with the added nuance that the witness for the cardinal of value $\aleph_1$ can be taken to be coanalytic.

More recently, Törnquist [21] has constructed a coanalytic mad family under weaker assumptions than $\mathsf{V}$=$\mathsf{L}$; namely, he shows that assuming there exists a $\Sigma^1_2$ mad family, then there exists a coanalytic mad family. His proof is purely combinatorial and simpler in application to the method mentioned above, and moreover has the advantage of being able to be applied in models of $\neg\mathsf{CH}$. More proofs resembling Törnquist's began appearing sporadically throughout the literature, for the cases of, for example, maximal independent families, maximal eventually different families, maximal orthogonal families of Borel probability measures, and more recently Hausdorff gaps (see [3], [8], [10], [18], respectively). A general framework and uniform presentation of these proofs can be found in [18].

In this paper we consider four cases of combinatorial sets of reals: Van Douwen families, maximal eventually different families of permutation, maximal cofinitary groups, and maximal ideal independent families. In each case we show that the existence of a $\Sigma^1_2$ such family implies the existence of a $\Pi^1_1$ family of the same size.

## 2. Van Douwen families

A family $F \subseteq {}^{\omega}\omega$ is called *eventually different* if $|f \cap g| < \omega$ for all distinct $f, g \in F$, i.e., there are only finitely many $n$ such that $f(n) = g(n)$. Such a family $F$ is *maximal* if it is maximal with respect to inclusion among all eventually different families; equivalently, for any $f \in \omega^{\omega}$ there exists $g \in F$ such that $|f \cap g| = \omega$. A strengthening of this notion of maximality is that of being Van Douwen; an eventually different family $F$ is *Van Douwen* if for any infinite partial function $f \in {}^{\omega}\omega$, there exists $g \in F$ with $|f \cap g| = \omega$. In other words, Van Douwen families are maximal eventually different families which are also maximal with respect to infinite partial functions.

Both $\mathsf{CH}$ and $\mathsf{MA}$ imply the existence of Van Douwen families. Zhang [23, Theorem 4.2] shows that under $\mathsf{CH}$, there exists a Cohen-indestructible Van Douwen family. Later, Raghavan [19] proved that there always is a Van Douwen family, answering a question by Van Douwen (thus its naming). Regarding definability, he also showed that Van Douwen families can never be analytic. This is in stark contrast to the situation for maximal eventually different families. There, Shelah and Horowitz [13] showed that there always exists a Borel maximal eventually different family of size $\mathfrak{c}$. However as Borel and analytic sets satisfy the perfect set property and maximal eventually different families cannot be countable, any Borel or analytic maximal eventually different family must always be of size $\mathfrak{c}$, and therefore it is of interest to ask about the definability of maximal eventually different families of size strictly less than $\mathfrak{c}$ in models of $\neg\mathsf{CH}$.

In [8] Fischer and Schrittesser constructed a maximal eventually different family indestructible by countably supported iteration or product of Sacks-forcing of any length, and they improve this result by showing a coanalytic such family exists in $\mathsf{L}$. Specifically, a Sacks indestructible maximal eventually different family can be constructed in $\mathsf{L}$ in a $\Sigma^1_2$ way, and then [8, Theorem 8] showed that the existence of a $\Sigma^1_2$ is equivalent to the existence of a coanalytic maximal eventually different family. Their argument follows the structure of Törnquists proof [21] that a $\Sigma^1_2$ mad family implies the existence of a $\Pi^1_1$ mad family. More specifically, they directly code a real into the function values of the functions composing the coanalytic maximal eventually different family. We will prove the analogous result for Van Douwen families; however, we need a different coding argument as their coding argument destroys the property of being Van Douwen.

**Theorem 1.** *If there is a $\Sigma^1_2$ Van Douwen family, then there is a $\Pi^1_1$ Van Douwen family of the same size.*

*Proof.* Define functions $\chi_0, \chi_1 : {}^{\omega}\omega \times {}^{\omega}2 \to {}^{\omega}\omega$

$$\chi_0(f, c)(n) := 2f(n) + c(n),$$
$$\chi_1(f, c)(n) := 2f(n) + 1 - c(n).$$

Now, assume that $F$ is a $\Sigma^1_2$ Van Douwen family. Further let $H \subseteq {}^{\omega}\omega \times {}^{\omega}2$ be $\Pi^1_1$ such that $F$ is the projection of $H$ to the first component. By uniformization we may assume that $H$ is the graph of a partial function. Let

$$G := \chi_0[H] \cup \chi_1[H].$$

We claim that $G$ is the desired Van Douwen family. So let $f_0 \in F$ and $c_0 := H(f_0)$. By construction we have for all $n \in \omega$

$$\chi_0(f_0, c_0)(n) \neq \chi_1(f_0, c_0)(n).$$

Similarly, for $f_1 \in F$ with $f_1 \neq f_0$ and $c_1 := H(f_1)$ we may choose $N \in \omega$ such that $f_0(n) \neq f_1(n)$ for all $n > N$. But then for all such $n > N$ and $i_0, i_1 \in 2$ we also have

$$\chi_{i_0}(f_0, c_0)(n) \neq \chi_{i_1}(f_1, c_1)(n).$$

Thus, all members of $G$ are eventually different. Now, let $g : A \to \omega$ be an infinite partial function. Then, we define $\hat{g} : A \to \omega$ by

$$\hat{g}(n) := \lfloor \frac{1}{2} g(n) \rfloor.$$

Since $F$ is Van Douwen, choose $f \in F$ and $B \in [A]^\omega$ such that $f \restriction B = \hat{g} \restriction B$. Let $c := H(f)$, then for all $n \in B$ there are $i, j \in 2$ with

$$g(n) = 2\hat{g}(n) + i = 2f(n) + i = \chi_j(f, c)(n).$$

Thus, either $g =^\infty \chi_0(f, c)$ or $g =^\infty \chi_1(f, c)$ as desired. As for the definability of $G$, we will show that for all $g \in {}^\omega\omega$,

$$g \in G \Leftrightarrow \exists (f, c) \in \Delta^1_1(g)[(f, c) \in H \wedge (\chi_0(f, c) = g \vee \chi_1(f, c) = g)],$$

which is a $\Pi^1_1$ definition by the Spector-Gandy theorem (see, for example, [17, Corollary 29.3]). Indeed, given any $g$, we have that $f = \lfloor \frac{g}{2} \rfloor$, where

$$\lfloor \frac{g}{2} \rfloor(n) = \begin{cases} \frac{g(n)}{2} & \text{if } g(n) \text{ even,} \\ \frac{g(n)-1}{2} & \text{if } g(n) \text{ odd.} \end{cases}$$

Clearly $g \mapsto \lfloor \frac{g}{2} \rfloor$ is a recursive function. Then we can define the reals $c(n) = i$ if and only if $g(n) \bmod 2 = i$, and $c'(n) = 1 - c(n)$. To check whether $(f, c) \in H$ or $(f, c') \in H$ are both $\Pi^1_1$, and checking $\chi_i(f, c) = g$ is Borel for each $i < 2$. This shows the $\Pi^1_1$-definability of $G$ above. □

**Corollary 2.** *It is consistent with $\mathfrak{c} \geq \aleph_2$ that there exists a coanalytic Van Douwen family of size $\aleph_1$.*

*Proof.* Repeat Zhang's construction of a Cohen-indestructible Van Douwen family $F$ in $\mathsf{L}$ , along an enumeration of the set of nice Cohen names for reals given by the $\Sigma^1_2$-definable wellorder $\leq_L$. After adding $\kappa$-many Cohen reals, where $\kappa \geq \aleph_2$ is a regular cardinal, $F$ is still a Van Douwen family, which has a $\Sigma^1_2$ definition by Shoenfield absoluteness. Apply the theorem above. □

## 3. Eventually different families of permutations

In this section we will consider eventually different families of permutations; these are eventually different families $F \subseteq S_\infty$, where $S_\infty$ denotes the set of permutations (i.e. bijections) of $\omega$. Such a family $F$ is maximal as an eventually different family of permutations if for any $g \in S_\infty$ there exists $f \in F$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$. The minimal size of a maximal eventually different family of permutations is denoted $\mathfrak{a}_p$. Constructions of models in which $\mathfrak{a}_e = \mathfrak{a}_p = \aleph_1 < \mathfrak{c} = \aleph_2$ with coanalytic witnesses can be found in [9]; it remains open whether $\mathfrak{a}_e = \mathfrak{a}_p$ is a theorem of $\mathsf{ZFC}$.

Witnesses for $\mathfrak{a}_p$ can be analytic, by results of Horowitz and Shelah, though one can ask about the definability of $\aleph_1$-sized maximal eventually different families of permutations in models with $\mathfrak{c} \geq \aleph_2$. In this section will show that a coanalytic witness for $\mathfrak{a}_p = \aleph_1$ already exists if there is a $\Sigma^1_2$ witness for $\mathfrak{a}_p = \aleph_1$. This is the optimal complexity of such a family in models with $\mathfrak{c} \geq \aleph_2$. The proof will make use of the following easy to prove graph theoretic fact:

**Fact 3.** *Every bipartite 2-regular graph decomposes into a disjoint union of cycles of even or infinite length and hence has a perfect matching by picking edges alternatingly.*

**Lemma 4.** *Assume $f : \omega \to \omega$ is 2-to-1, i.e. every $n \in \omega$ has exactly two preimages. Then there is a function $i : \omega \to 2$ such that the function $g : \omega \to \omega$ defined by $g(n) := f(2n + i(n))$ is a bijection.*

*Proof.* Consider the following bipartite graph $H$ (with possible multi-edges):

(i) We have countably many left $\{L_n \mid n \in \omega\}$ and right $\{R_n \mid n \in \omega\}$ nodes,

(ii) For each $n \in \omega$ we have an edge $e_n$ between $L_{\lfloor \frac{n}{2} \rfloor}$ and $R_{f(n)}$.

By construction, every $L_n$ has degree 2. As $f$ is 2-to-1 the same holds for the $R_n$, i.e. $H$ is 2-regular. Hence, $H$ has a perfect matching $P$ by the fact above. Now, we define for $n \in \omega$

$$i(n) := \begin{cases} 0 & \text{if } e_{2n} \in P, \\ 1 & \text{if } e_{2n+1} \in P. \end{cases}$$

Note, for every $n \in \omega$ that $e_{2n}$ and $e_{2n+1}$ are the only edges incident to $L_n$. Thus, exactly one of these cases above occurs as we have a perfect matching. It is also easy to see that $g$ will then be bijective: If $g$ was not injective, then $P$ would not be a matching, and if $g$ was not surjective, then $P$ would not be perfect. $\square$

**Theorem 5.** *If there is a $\Sigma^1_2$ maximal eventually different family of permutations, then there is a $\Pi^1_1$ maximal eventually different family of permutations of the same size.*

*Proof.* Define functions $\chi_0, \chi_1 : S_\infty \times {}^\omega 2 \to S_\infty$ for $n \in \omega$ by

$$\begin{aligned} \chi_0(f, c)(2n) &:= 2f(n) + c(n), \\ \chi_0(f, c)(2n + 1) &:= 2f(n) + 1 - c(n), \end{aligned}$$

$$\begin{aligned} \chi_1(f, c)(2n) &:= 2f(n) + 1 - c(n), \\ \chi_1(f, c)(2n + 1) &:= 2f(n) + c(n). \end{aligned}$$

Now, assume that $F$ is a $\Sigma^1_2$ maximal family of permutations. Further let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be $\Pi^1_1$ such that $F$ is the projection of $H$ to the first component. By uniformization we may assume that $H$ is the graph of a partial function. Let

$$\hat{F} := \chi_0[H] \cup \chi_1[H].$$

We claim that $\hat{F}$ is the desired maximal eventually different family of permutations. Let $f \in S_\infty$, $c \in {}^\omega 2$ and $i \in 2$. First we show that $\chi_i(f,c) \in S_\infty$. Assume for $n_0, n_1 \in \omega$ and $j_0, j_1 \in 2$ we have

$$\chi_i(f,c)(2n_0 + j_0) = \chi_i(f,c)(2n_1 + j_1)$$

By construction, this implies that $f(n_0) = f(n_1)$. But $f$ is injective, so also $n_0 = n_1$. Again, by construction of $\chi_i(f,c)$ we also obtain $j_0 = j_1$, i.e. $\chi_i(f,c)$ is injective. For surjectivity, let $m \in \omega$ and $j \in 2$. Since $f$ is surjective, choose $n$ such that $f(n) = m$. But then either

$$\chi_i(f,c)(2n) = 2f(n) + j = 2m + j$$

or we have

$$\chi_i(f,c)(2n+1) = 2f(n) + j = 2m + j.$$

Hence $\chi_i(f,c)$ is surjective. Next, let $f_0 \in F$ and $c_0 := H(f_0)$. By construction we have for all $n \in \omega$

$$\chi_0(f_0,c_0)(n) \neq \chi_1(f_0,c_0)(n).$$

Similarly, for $f_1 \in F$ with $f_1 \neq f_0$ and $c_1 := H(f_1)$ we may choose $N \in \omega$ such that $f_0(n) \neq f_1(n)$ for all $n > N$. But then for all such $n > N$ and $i_0, i_1 \in 2$ we also have

$$\chi_{i_0}(f_0,c_0)(n) \neq \chi_{i_1}(f_1,c_1)(n).$$

Thus, all members of $\hat{F}$ are eventually different. Now, towards maximality of $\hat{F}$ let $\hat{g} \in S_\infty$. Choose a function $i : \omega \to 2$ such that $g : \omega \to \omega$ defined by

$$g(n) := \lfloor \frac{\hat{g}(2n + i(n))}{2} \rfloor$$

is a bijection. This is possible by the previous lemma as the function $\lfloor \frac{\hat{g}(n)}{2} \rfloor$ is 2-to-1. By maximality of $F$, we may choose $f \in F$ and $A \in [\omega]^\omega$ such that $g \restriction A = f \restriction A$. Let $c := H(f)$ and $n \in A$. Then there are $j, k \in 2$ so that

$$\hat{g}(2n + i(n)) = 2g(n) + j = 2f(n) + j = \chi_k(f,c)(2n + i(n)).$$

Thus, either $\hat{g} =^\infty \chi_0(f,c)$ or $\hat{g} =^\infty \chi_1(f,c)$ as desired. As before Spector-Gandy shows that $\hat{F}$ is $\Pi^1_1$ as for fixed $i \in 2$ we can compute $(f,c)$ from $\chi_i(f,c)$. $\square$

Before we move on to maximal cofinitary groups, we discuss the minimal complexity of a maximal family of permutations. For $\mathfrak{a}_e$ Schrittesser [20] showed that there is a $\Pi^0_1$, i.e. a closed maximal eventually different family. Similarly, for $\mathfrak{a}_g$ Mejak and Schritesser showed that there is a $\Pi^0_1$ set freely generating a maximal cofinitary group. Thus, the whole group has complexity $\Sigma^0_2$. Moreover, this group is not only maximal as a cofinitary group, but also maximal as an eventually different family of permutations (see [16, Proposition 2.13])). Hence, they also proved that there is a $\Sigma^0_2$ witness for $\mathfrak{a}_p$, however to the knowledge of the authors it is not known what the minimal complexity for a maximal eventually different family of permutations is.

**Question 6.** Is there a $\Pi^0_1$ maximal eventually different family of permutations?

In the same way Schrittesser obtained a $\Pi^0_1$ maximal eventually different family, one might assume that our proof above may be used to obtain an analogous statement to [20, Lemma 4.1]. However, with our coding above we only get the following:

**Lemma 7.** *Let $0 < \xi < \omega_1$. If there is a $\Pi^0_{\xi+2}$ maximal eventually different family of permutations, then there is a $\Pi^0_{\xi+1}$ maximal eventually different family of permutations.*

*Proof.* Adapt [20, Lemma 4.1] using the coding above. □

Note the extra assumption of $0 < \xi$, so with this lemma we can only obtain a $\Pi^0_2$ witness for $\mathfrak{a}_p$. Essentially, this is due to the coding presented above coding elements of Cantor space instead of Baire space. Hence, in order to define the maximal eventually different family of permutations of lower complexity one needs to express that the sequence of (non-)flips encodes a sequence of natural numbers. However, to this end we need to require that the sequence of (non-)flips is not eventually constant. But this is a $\Pi^0_2$ statement, thus requiring the extra assumption.

## 4. Cofinitary groups

Next, we will consider maximal cofinitary groups. If an eventually different family of permutations $G$ is also a group with respect to concatenation, then we call it a *cofinitary group*. Equivalently, every element of $G$ is either the identity or only has finitely many fix-points. $G$ is *maximal* if its maximal with respect to inclusion among all cofinitary groups. The minimal cardinality of a maximal cofinitary group is uncountable and denoted with $\mathfrak{a}_g$; again no known relations or the absence thereof between $\mathfrak{a}_e, \mathfrak{a}_p$ and $\mathfrak{a}_g$ are known.

In terms of definability it is often easier to obtain a definable generating set for a cofinitary group. We say that $F$ *generates* $G$ if $\langle F \rangle = G$, where $\langle F \rangle$ is the group generated by $F$. If there are no relations among the generators in $F$, we say that $F$ *freely generates* $G$. For example, it was first shown by Gao and Zhang [11] that in $\mathsf{L}$ there is a $\Pi^1_1$ generating set for a maximal cofinitary group, before Kastermans [15] showed that indeed the entire group can be $\Pi^1_1$. Later, Horowitz and Shelah [14] showed that there always is a Borel maximal cofinitary group. More recently, Fischer, Schrittesser and the second author [7] showed that in $\mathsf{L}$ there is a $\Pi^1_1$ cofinitary group which is indestructible by various different tree forcings preserving $\mathrm{non}(\mathcal{M})$. They employed an intricate coding argument, where information is coded into the lengths orbits and into the amount of orbits of a certain length.

Here, we use a simpler coding technique to obtain a result for cofinitary groups similar to the previous sections. However, with our methods we can only get a result for generating sets of cofinitary groups and we need to additionally assume that the maximal cofinitary group is freely generated and also maximal as an eventually different family of permutations. This seems like a strong extra assumption, but indeed most cofinitary groups constructed in the natural way, satisfy these extra assumptions. In particular, the next theorem provides a different way to show that many $\mathsf{L}$-extensions have a coanalytic generating set for a maximal cofinitary group.

**Theorem 8.** *If there is a $\Sigma^1_2$ family freely generating a maximal cofinitary group, which is also maximal as an eventually different family of permutations, then there is a $\Pi^1_1$ family generating a maximal cofinitary group, which is also maximal as an eventually different family of permutations.*

*Proof.* As in Theorem 5 define $\chi_0, \chi_1 : S_\infty \times {}^\omega 2 \to S_\infty$ for $n \in \omega$ by

$$\chi_0(f,c)(2n) := 2f(n) + c(n),$$
$$\chi_0(f,c)(2n+1) := 2f(n) + 1 - c(n),$$

$$\chi_1(f,c)(2n) := 2f(n) + 1 - c(n),$$
$$\chi_1(f,c)(2n+1) := 2f(n) + c(n).$$

This time, assume that $F$ is a $\Sigma^1_2$ set freely generating the maximal cofinitary group $\Gamma := \langle F \rangle$. Further let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be $\Pi^1_1$ such that $F$ is the projection of $H$ to the first component. By uniformization we may assume that $H$ is the graph of a partial function. Let

$$\hat{F} := \chi_0[H] \cup \chi_1[H].$$

By the arguments in the previous section $\hat{F}$ is a family of permutations and is $\Pi^1_1$. It remains to show that for $\hat{\Gamma} := \langle \hat{F} \rangle$ we have

(1) $\hat{\Gamma}$ is cofinitary,
(2) $\hat{\Gamma}$ is maximal as an eventually different family of permutations.

From now on we consider the partition of $\omega$ given by the pairs $B_n := \{2n, 2n+1\}$ for $n \in \omega$. We need the following lemmata:

**Lemma 9.** *For every $\hat{g} \in \hat{\Gamma}$ there is a unique $g \in \Gamma$ such that for all $n \in \omega$*

$$\hat{g}[B_n] = B_{g(n)}.$$

*Moreover, the assignment $\Psi : \hat{\Gamma} \to \Gamma$ given by $\hat{g} \mapsto g$ is a surjective group homomorphism with $\Psi(\chi_i(f, H(f))) = f$ for all $f \in F$ and $i \in 2$.*

*Proof.* For uniqueness, suppose $g, h \in \Gamma$ satisfy $\hat{g}[B_n] = B_{g(n)} = B_{h(n)}$. Since the $B_n$'s are disjoint this implies $g = h$, so it suffices to prove existence. To this end, for $i \in 2$, $f \in F$ and $c := H(f)$ by definition of $\chi_i$ we clearly have that

$$\chi_i(f,c)[B_n] = B_{f(n)}.$$

Similarly, it is easy to see that also for the inverse $\chi_i(f,c)^{-1}$ we have

$$\chi_i(f,c)^{-1}[B_n] = B_{f^{-1}(n)}.$$

Thus, the required $g \in \Gamma$ exists for all generators and inverses thereof in $\hat{\Gamma}$. We prove the general case by induction on the length of $x$, so let $\hat{g} = \hat{x}_1 \ldots \hat{x}_k$ be a reduced word with letters in $\hat{F}^{\pm 1}$. If $k = 0$ then $\hat{g} = \mathrm{id}_\omega$ and $\mathrm{id}_\omega[B_n] = B_n = B_{\mathrm{id}_\omega(n)}$ with $\mathrm{id}_\omega \in \Gamma$. Now, let $k > 0$, $\hat{h}_1 = \hat{x}_1$ and $\hat{h}_2 = \hat{x}_2 \ldots \hat{x}_k$. By induction, there are $h_1, h_2 \in \Gamma$ such that $\hat{h}_1[B_n] = B_{h_1(n)}$ and $\hat{h}_2[B_n] = B_{h_2(n)}$ for all $n \in \omega$. Then, for $n \in \omega$ we compute

$$\hat{g}[B_n] = \hat{h}_1[\hat{h}_2[B_n]] = \hat{h}_1[B_{h_2(n)}] = B_{h_1(h_2(n))} = B_{(h_1 \circ h_2)(n)},$$

proving the existence of the desired $g := h_1 \circ h_2$. This computation also proves the homomorphism property of $\Psi$. Furthermore, we obtain

$$\Psi[\hat{\Gamma}] = \Psi[\langle \hat{F} \rangle] = \langle \Psi[\hat{F}] \rangle = \langle F \rangle = \Gamma,$$

so $\Psi$ is surjective. $\square$

**Lemma 10.** *Let $\tau : \omega \to \omega$ be the flip map defined for $n \in \omega$ by*

$$\tau(2n) := 2n+1 \quad \textit{and} \quad \tau(2n+1) := 2n.$$

*Then $\tau \in \hat{\Gamma}$ is central in $\hat{\Gamma}$ and we have* $\ker(\Psi) = \{\mathrm{id}_\omega, \tau\}$.

*Proof.* First, note that $\tau^2 = \mathrm{id}_\omega$ and for every $f \in F$, $i \in 2$ and $c := H(f)$ we have

$$\chi_0(f, c)\chi_1(f, c)^{-1} = \tau \quad \text{and} \quad \chi_1(f, c)^{-1}\chi_0(f, c) = \tau.$$

This shows that $\tau \in \hat{\Gamma}$ and that $\tau$ commutes with all generators of $\hat{\Gamma}$. Thus, $\tau$ is central in $\hat{\Gamma}$. Further, $\Psi(\tau) = \mathrm{id}_\omega$, so $\ker(\Psi) \supseteq \{\mathrm{id}_\omega, \tau\}$. Now, let $\hat{g} \in \ker(\Psi)$. Write $\hat{g}$ as a word in letters from $\hat{F}^{\pm 1}$. Using $\chi_1^{\pm 1} = \tau\chi_0^{\pm 1} = \chi_0^{\pm 1}\tau$, centrality of $\tau$ and $\tau^2 = \mathrm{id}_\omega$, replace each occurrence of $\chi_1$ by $\chi_0$ and move the $\tau$ to the front to obtain

$$\hat{g} = \tau^i \hat{h},$$

where $i \in 2$ and $\hat{h}$ is a word in the alphabet $\{\chi_0(f, H(f))^{\pm 1} \mid f \in F\}$. Remember $\hat{g}, \tau \in \ker(\Psi)$, so apply $\Psi$ to obtain

$$\mathrm{id}_\omega = \Psi(\hat{g}) = \Psi(\tau^i \hat{h}) = \Psi(\tau^i)\Psi(\hat{h}) = \Psi(\hat{h}).$$

But by Lemma 9 $\Psi$ is a homomorphism which maps $\chi_0(f, H(f))$ to $f$, so $\Psi(\hat{h})$ is the corresponding word in the generators $F^{\pm 1}$ of $\Gamma$. But $\Gamma = \langle F \rangle$ is freely generated, so the equation above implies that $\hat{h} = \mathrm{id}_\omega$ Consequently, $\hat{g} = \tau^i \in \{\mathrm{id}_\omega, \tau\}$. $\square$

**Proposition 11.** *$\hat{\Gamma}$ is cofinitary.*

*Proof.* Let $\hat{g} \in \hat{\Gamma} \setminus \{\mathrm{id}_\omega\}$. If $\hat{g} \in \ker(\Psi)$, then by the previous lemma we can only have $\hat{g} = \tau$ which has no fixpoints. Thus, we may assume that $g := \Psi(x) \in \Gamma \setminus \{\mathrm{id}_\omega\}$. Let $k \in \omega$ be a fixpoint of $\hat{h}$, say $k \in B_n$, then by Lemma 9 we have $B_n = B_{g(n)}$. Thus, $n = g(n)$, so we get

$$\mathrm{fix}(\hat{g}) \subseteq \bigcup_{n \in \mathrm{fix}(g)} B_n.$$

But $g$ is cofinitary and each $B_n$ has size 2, so $|\mathrm{fix}(\hat{g})| \leq 2\,|\mathrm{fix}(g)| < \infty$. $\square$

**Proposition 12.** *$\hat{\Gamma}$ is maximal as an eventually different family of permutations.*

*Proof.* Let $\hat{h} \in S_\infty$ and by Lemma 4 choose $i : \omega \to 2$, so that

$$h(n) := \lfloor \frac{g(2n + i(n))}{2} \rfloor$$

is a bijection. By maximality of $\Gamma$ as an eventually different family of permutations there is $g \in \Gamma$ and $A \in [\omega]^\omega$ such that $h \restriction A = g \restriction A$. Let $\hat{h}$ be the corresponding word in $\Gamma'$, where every occurrence of $f^{\pm 1}$ is replaced by $\chi_0(f, H(f))^{\pm 1}$. Thus, we have $\Psi(\hat{g}) = g$, so as before for every $n \in A$ there are $i, j \in 2$ such that

$$\hat{h}(2n + i_n) = 2h(n) + i = 2g(n) + i = \tau^j \hat{g}(2n + i_n).$$

But this implies either $\hat{g} =^\infty \hat{h}$ or $\hat{g} =^\infty \tau\hat{h}$ as desired. □

**Remark 13.** The generated group is not free but by the considerations above its isomorphism type is given by the product $\mathbb{Z}/2 \times F$, where $(1, \mathrm{id}_\omega)$ corresponds to the element $\tau$.

**Question 14.** What about non-freely generated groups or whole groups?

## 5. Ideal independent families

A family $\mathcal{I} \subseteq [\omega]^\omega$ is *ideal independent* if for all $F \in [\mathcal{I}]^{<\omega}$ and $a \in \mathcal{I} \setminus F$, it is not the case that $a \subseteq^* \bigcup F$. An ideal independent family $\mathcal{I}$ is *maximal ideal independent* if $\mathcal{I}$ is maximal with respect to inclusion. Equivalently, for every $b \in [\omega]^\omega$ there exists a finite $F \subseteq \mathcal{I}$ such that one of the following occurs:

- $b \subseteq^* \bigcup F$, or
- there is $a \in \mathcal{I} \setminus F$ such that $a \subseteq^* b \cup \bigcup F$.

The cardinal $\mathfrak{s}_{mm}$ is defined as the minimum size of a maximal ideal independent family. Investigations of the relations between $\mathfrak{s}_{mm}$ and other cardinal invariants can be found in [4] and [1]. Clearly maximal ideal independent families can be obtained using the Axiom of Choice, however as of now the minimal projective complexity of such a family is unknown. The theorem below gives an upper bound of $\Pi^1_1$.

**Theorem 15.** *If there exists a $\Sigma^1_2$ maximal ideal-independent family, then there exists a $\Pi^1_1$ maximal ideal-independent family of the same size.*

*Proof.* Let $\mathcal{I}$ be a $\Sigma^1_2$ maximal ideal independent family, and let $H \subseteq [\omega]^\omega \times [\omega]^\omega$ be a coanalytic set such that $x \in \mathcal{I}$ if and only if there exists $c \in [\omega]^\omega$ with $(x, c) \in H$; by uniformization, we can assume $H$ is the graph of a function. Define a function $\chi : [\omega]^\omega \times [\omega]^\omega \to [\omega]^\omega$ and $z \in [\omega]^\omega$ by

$$\begin{aligned} \chi(x, c) &:= 3x \cup 3c + 1 && = \{3n \mid n \in x\} \cup \{3n + 1 \mid n \in c\}, \\ z &:= 3\omega + 1 \cup 3\omega + 2 && = \{3n + 1 \mid n \in \omega\} \cup \{3n + 2 \mid n \in \omega\}. \end{aligned}$$

Let

$$\mathcal{J} := \chi[H] \cup \{z\}.$$

We will show $\mathcal{J}$ is maximal ideal independent. To see it is ideal independent, let $F \in [\mathcal{J}]^{<\omega}$ be arbitrary and let $x \in \mathcal{J} \setminus F$. First, suppose $x \neq z$ and $z \notin F$. Let $(a, c) \in ([\omega]^\omega)^2$ be such that $\chi(a, c) = x$. Then $x \setminus \bigcup F$ must be infinite, as it contains the set $a \setminus \bigcup \mathrm{proj}_0[\chi^{-1}[F]]$, where $\mathrm{proj}_0 : [\omega]^\omega \times [\omega]^\omega \to [\omega]^\omega$ is the projection onto the first coordinate. Now suppose $x = z$. Then $z \setminus \bigcup F$ is infinite, as it contains the set $\{3n + 2 \mid n \in \omega\} \subseteq z$. Lastly, suppose $x \neq z$ and $z \in F$; by the first case, it suffices to consider the case $F = \{z\}$. But clearly, the set $\{3n \mid n \in a\} \subseteq x$ is an infinite subset disjoint from $z$.

Towards maximality, let $b \in [\omega]^\omega$, and consider $b_0 := \{n \in \omega \mid 3n \in b\}$. By maximality of $\mathcal{I}$, there exists $F \in [\mathcal{I}]^{<\omega}$ such that one of the following occurs:

(1) $b_0 \subseteq^* \bigcup F$, or
(2) there exists $x \in \mathcal{I} \setminus F$ such that $x \subseteq^* b_0 \cup \bigcup F$.

Suppose we are in the first case. Then $b \subseteq^* \{z\} \cup \bigcup \chi[(F \times [\omega]^\omega) \cap H]$, since on the one hand $b \cap 3\omega$ is almost covered by $\chi[(F \times [\omega]^\omega) \cap H]$, and on the other hand,

$$b \setminus 3\omega = \{3n+1 \in b\} \cup \{3n+2 \in b\} \subseteq z.$$

Otherwise, fix $x \in \mathcal{I}$ as in case (2), and let $c$ be such that $(x, c) \in H$. Then $\chi(x, c)$ is an element of $\mathcal{J}$ such that $\chi(x, c) \subseteq^* b \cup \bigcup(\chi[(F \times [\omega]^\omega) \cap H] \cup \{z\})$, since $\{3n+1 \mid n \in c\} \subseteq z$.

We claim that $\mathcal{J}$ is coanalytic. First, note that the set $\{z\}$ is clearly $\Delta^1_1$-definable. Next, we have that the set $\chi[H]$ is $\Pi^1_1$, as it is defined by

$$x \in \chi[H] \Leftrightarrow \exists a, c \in \Delta^1_1(x)[(a, c) \in H \wedge \chi(a, c) = x].$$

Indeed, given $x \in [\omega]^\omega$, $a$ is reconstructible by $x$ as the set $\{n \in \omega \mid 3n \in x\}$. Similarly, also $c$ is reconstructible from $x$. Then, being the union of two coanalytic sets, $\mathcal{J}$ is coanalytic. $\square$

An ultrafilter $\mathcal{U}$ is called a *p-point* if for any countable $\mathcal{F} \subseteq \mathcal{U}$, there exists $Y \in \mathcal{U}$ such that $Y \subseteq^* X$ for every $X \in \mathcal{F}$. Recently, Bardyla, Cancino-Manriquez, Fischer, and Switzer have defined the notion of *$\mathcal{U}$-encompassing* ideal independent family, where $\mathcal{U}$ is an ultrafilter on $\omega$ (see [1, Definition 5.1]). When such $\mathcal{U}$ is a p-point and under some additional assumptions, this strengthening of the maximality condition for ideal independent families isolates a subclass of maximal ideal independent families which are indestructible by any proper, $\omega^\omega$-bounding, p-point preserving forcing notion. Moreover they show that under $\mathsf{CH}$, for any p-point $\mathcal{U}$ there exists a $\mathcal{U}$-encompassing ideal independent family with the required additional assumptions for indestructibility [1, Theorem 5.2]. Thus, adapting their construction in $\mathsf{L}$ and using the $\Sigma^1_2$-definable wellorder of the reals yields a $\Sigma^1_2$-definable maximal ideal independent family which remains maximal under a broad class of proper forcing notions. One particular instance of a proper, $\omega^\omega$-bounding forcing notion which preserves p-points is the construction of a $\Delta^1_3$ wellorder of the reals in a model of $\mathfrak{c} = \aleph_2$ given by Fischer and Friedman in 2010 [6]; see also [2]. Using these observations and Theorem 15 we obtain the following:

**Corollary 16.** *Consistently, we have have $\aleph_1 = \mathfrak{s}_{mm} < \mathfrak{c} = \aleph_2$ together with a coanalytic witness for $\mathfrak{s}_{mm}$ and a $\Delta^1_3$ total wellorder of the reals.*

## References

[1] Serhii Bardyla, Jonathan Cancino-Manríquez, Vera Fischer, and Corey Bacal Switzer. Filters, ideal independence and ideal Mrówka spaces. *Transactions of the American Mathematical Society*, 378(8):5423–5439, 2025.

[2] Jeffrey Bergfalk, Vera Fischer, and Corey Bacal Switzer. Projective well orders and coanalytic witnesses. *Annals of Pure and Applied Logic*, 173(8):103135, 2022.

[3] Jörg Brendle, Vera Fischer, and Yurii Khomskii. Definable Maximal Independent Families. *Proceedings of the American Mathematical Society*, (147):3547–3557, 2019.

[4] Jonathan Cancino, Osvaldo Guzmán, and Arnold W. Miller. Ideal Independent Families and the Ultrafilter Number. *The Journal of Symbolic Logic*, 86(1):128–136, 2021.

[5] Paul Erdős, Kenneth Kunen, and R. Daniel Mauldin. Some additive properties of sets of real numbers. *Fundamenta Mathematicae*, 113(3):187–199, 1981.

[6] Vera Fischer and S. D. Friedman. Cardinal characteristics and projective well-orders. *Annals of Pure and Applied Logic*, 161:916–922, 2010.

[7] Vera Fischer, Lukas Schembecker, and David Schrittesser. Tight cofinitary groups. *Annals of Pure and Applied Logic*, 176(6), 2025.

[8] Vera Fischer and David Schrittesser. A Sacks indestructible co-analytic maximal eventually different family. *Fundamenta Mathematicae*, 252:179–201, 2021.

[9] Vera Fischer and Corey Switzer. Tight eventually different families. *Journal of Symbolic Logic*, 2023.

[10] Vera Fischer and Asger Törnquist. A co-analytic maximal set of orthogonal measures. *The Journal of Symbolic Logic*, 75(4):1403–1414, December 2010.

[11] Su Gao and Yi Zhang. Definable sets of generators in maximal cofinitary groups. *Advances in Mathematics*, 217(2):814–832, 2008.

[12] Kurt Gödel. Consistency-proof for the generalized continuum-hypothesis. *Proceedings of the National Academy of Sciences*, 25(4):220–224, 1939.

[13] Haim Horowitz and Saharon Shelah. A Borel maximal eventually different family. *Annals of Pure and Applied Logic*, 175(1), 2024.

[14] Haim Horowitz and Saharon Shelah. A Borel maximal cofinitary group. *The Journal of Symbolic Logic*, 90(2):808–821, 2025.

[15] Bart Kastermans. Isomorphism types of maximal cofinitary groups. *The Bulletin of Symbolic Logic*, 15(3):300–319, 2009.

[16] Severin Mejak and David Schrittesser. Definability of maximal cofinitary groups, 2022.

[17] Arnold W. Miller. Infinite Combinatorics and Definability. *Annals of Pure and Applied Logic*, 41(2):179–203, 1989.

[18] Julia Millhouse. Reducing Projective Complexity: an Overview. volume 2315 of *RIMS Kôkyûroku*. Kyoto University, 2024.

[19] Dilip Raghavan. There is a Van Douwen family. *Transactions of the American Mathematical Society*, 362(11):5879–5891, 2010.

[20] David Schrittesser. On Horowitz and Shelah's Borel maximal eventually different family. volume 2164 of *RIMS Kôkyûroku*, pages 61–84. Kyoto University, 2017.

[21] Asger Törnquist. $\Sigma^1_2$ and $\Pi^1_1$ mad families. *J. Symbolic Logic*, 78(4), 2013.

[22] Zoltán Vidnyánszky. Transfinite inductions producing coanalytic sets. *Fundamenta Mathematicae*, 224(2):155–174, 2014.

[23] Yi Zhang. Towards a Problem of E. Van Douwen and A. Miller. *Mathematical Logic Quarterly*, 45(2):183–188, 1999.